From Students' Problem-Solving Strategies to Connections in Fractions

Many elementary teachers find it challenging to help their students understand mathematical concepts and make connections to other content. Concepts related to fractions can be particularly difficult for some students to grasp. This article presents strategies that children used to solve a fraction problem, in order to give some insight into how students think about division and fractions. It also illustrates how teachers can use these strategies to help students establish connections among different concepts related to fractions.

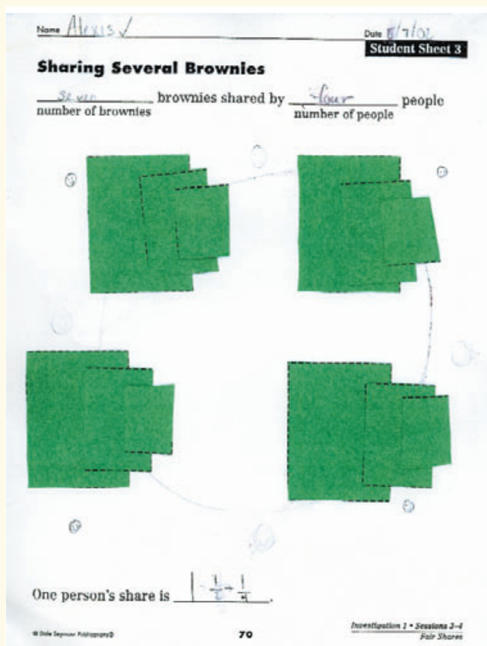
Erika Klein, a third-grade teacher, posed the following problem to her students:

> Tonight my mom, dad, grandma, and I will sit down to dinner. There will be seven brownies for dessert. How can we share the brownies so that everyone has the same amount? I really want to make sure that my dad, who can be very sneaky when it comes to brownies, does not get a bigger share. (Adapted from Tierney and Berle-Carman 1995)

Before posing this problem to her class, Ms. Klein gave each student a big sheet of paper with seven colored rectangles of the same size. The rectangles represented the seven brownies the students had to share. She also gave the students scissors, glue, and a piece of paper to display a model of the processes they used to share the fractions and their solutions. Students worked on the problem on their own, often sharing some of their ideas with their partners as they developed their solutions. As part of their solution strategy, many students depicted four people on the blank sheet of paper or divided it into four sections. Students started to put brown-

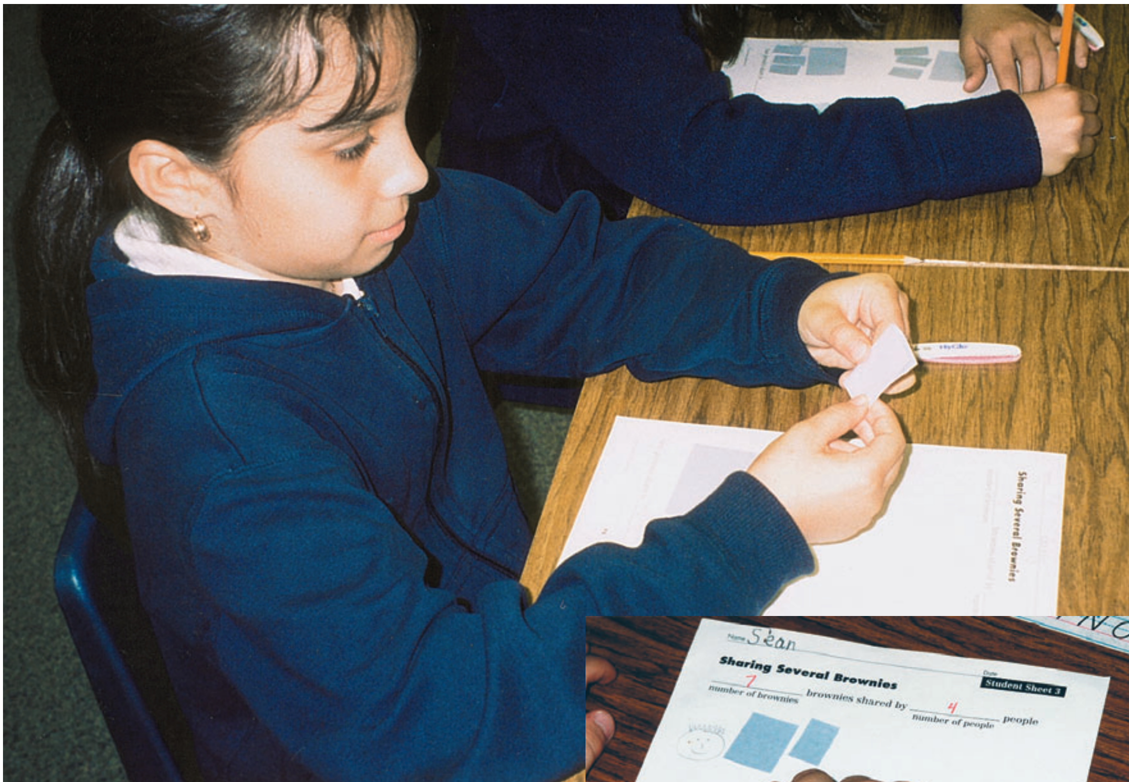
Figure 1

Alexis's solution



By Alfinio Flores and Erika Klein

Alfinio Flores, alfinio@asu.edu, teaches graduate and undergraduate mathematics education courses at Arizona State University. He enjoys visiting classrooms and observing students' ways to solve problems. Erika Klein, Sammy20330@aol.com, teaches third grade at Madison Park School. She likes to provide mathematical tasks to students so they can use their own strategies and share with the rest of the class.



ies and parts of brownies in the corresponding section, making sure that each one received the same amount.

The children used different strategies to solve the brownie problem. Alexis gave one brownie to each person. Then she cut one of the remaining brownies in half and gave one piece each to two of the people. She then cut another brownie into two equal parts and gave one-half to each of the other two people. The remaining brownie was cut into four equal parts, and each person received one-fourth. She glued the pieces and reported her answer as 1 1/2 + 1/4 (see **fig. 1**). Theodora used the same strategy and labeled the pieces with their corresponding values: 1 whole, 1/2, and 1/4. She reported her answer as 1 3/4 (see **fig. 2**). Other students recorded their answers as "1, 1/2, 1/4," or as "1 + 1/2 + 1/4."

Slightly different strategies also led to 1 + 1/2 + 1/4. One student solved the problem by giving one whole to each person. Then she broke the remaining three wholes into six halves and gave one to each person. She then broke the remaining two halves into fourths and gave one to each person.

Natasha also started by giving one brownie to each person. Then she divided one of the remaining brownies into four equal parts and gave 1/4 to each person. She repeated the same process with the other two brownies. She displayed her answer as 1 3/4 (see **fig. 3**). Another student used a very similar strategy, but after giving one brownie to each person, she cut all three remaining brownies into four parts.

Connor decided to cut each of the seven brownies into four equal parts so that each of the parts was 1/4. He realized that he had 28 such pieces, divided 28 by 4, and gave 7 pieces to each person (see **fig. 4**). He reported his solution as 7/4.

After most students had solved the problem and recorded their solution on paper, Ms. Klein asked several students to share their strategies and solutions with the rest of the class. As students described their solutions orally, Ms. Klein modeled their processes on the overhead projector. The teacher represented seven brownies and four people. Students explained what they had done with each brownie or fractional part and Ms. Klein mod-

Figure 2

Theodora's solution

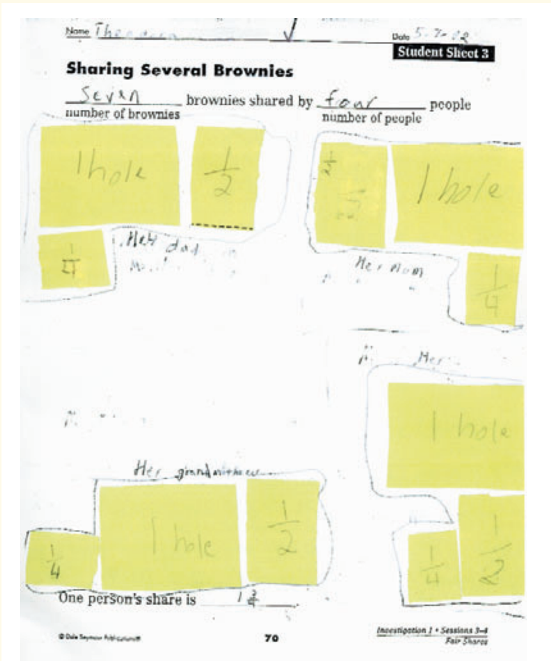
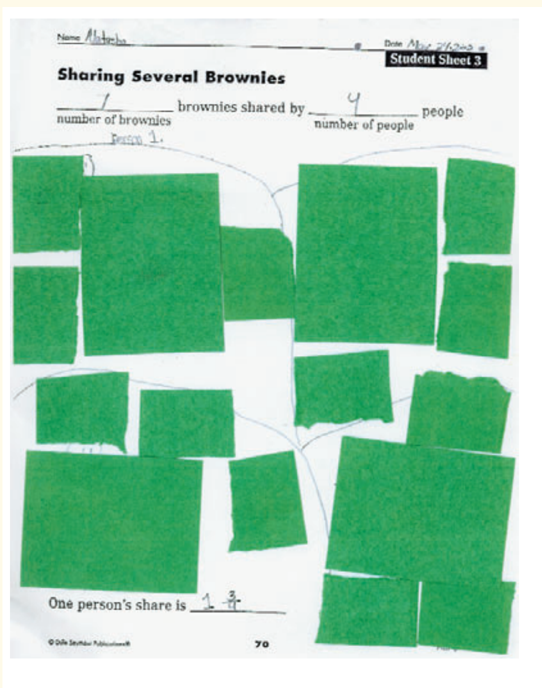


Figure 3

Natasha's solution



eled that step on the overhead projector, placed the corresponding piece next to each person, and labeled each piece with the corresponding fraction. Once a brownie had been shared, it was crossed out from the pool (see **fig. 5**). Other students went to the front of the classroom with their display sheets and explained their strategies.

Helping Make Connections

Third- or fourth-grade activities such as the one described in this article can set the stage for making explicit many important mathematical connections. One connection is the relationship between the division of whole numbers and fractions. Often, students conceptualize division of whole numbers and fractions as separate and unrelated topics. A setting in which the units can be further subdivided can help students make the connection. Consider how different the situation would be if the objects to be shared could not be subdivided: A remainder, rather than a fraction, would be a more natural part of the response. For example, if 9 marbles were to be divided among 4 children, each child would get 2 marbles and 1 marble would remain; it would not be possible for the children to cut 1/4 of a marble.

Some children will notice that the same numbers appear in the stated division problem 7 ÷ 4 and in the answer, 7/4. Kristin, a fifth grader, had written the values 1, 1/2, and 1/4 in her answer. She then converted everything to fourths and found that each person had received 7/4. She looked at the numbers in the answer and in the statement of the problem. She underlined the 7 and the 4 of the original problem 7 ÷ 4 and expressed her amazement as she pointed at the 7 and the 4: "It is just weird that it came equal to this and this." Some students will notice the pattern after a few examples; some will need explicit guidance to establish the connection. Teachers can help students see the relationship by guiding them to rewrite their answers in several ways, which will bring the connection to the forefront. A child who writes the answer of a problem such as 5 ÷ 4 as 1 1/4 may see the connection more readily if encouraged to write the answer also as 5/4.

People who know the connection between division and fractions often switch back and forth between the two concepts with ease, and sometimes without even noticing. The slanted fraction bar often indicates division. For example, many calculators use the slanted fraction bar to indicate the division key. Some calculators that use the symbol ÷ on the division key use the slanted bar on

the display. Students need time and opportunity to establish that connection, however. The relationship between 3 ÷ 4 and 3/4 will not be automatic for most children, nor should we assume that it is a natural connection. Some children will even resist using fractions to convey a division problem. A student in another classroom stated vehemently, "We are not talking about fractions, we are talking about dividing" (Toluk 1999, p. 182).

Another connection that can be made using the brownie problem as a springboard is that of equivalent fractions. Ms. Klein displayed three answers obtained using different strategies: 1 + 1/2 + 1/4; 1 3/4; and 7/4. The class discussed the different answers and agreed that they were all equal. Students argued that in each case, seven brownies were divided equally among four people, so the answers represented the same amount even though they were written differently. One student stated, "Just because you have different numbers does not mean they [the fractions] are different." Ms. Klein summarized the equivalencies on the overhead projector by writing the following three equations:

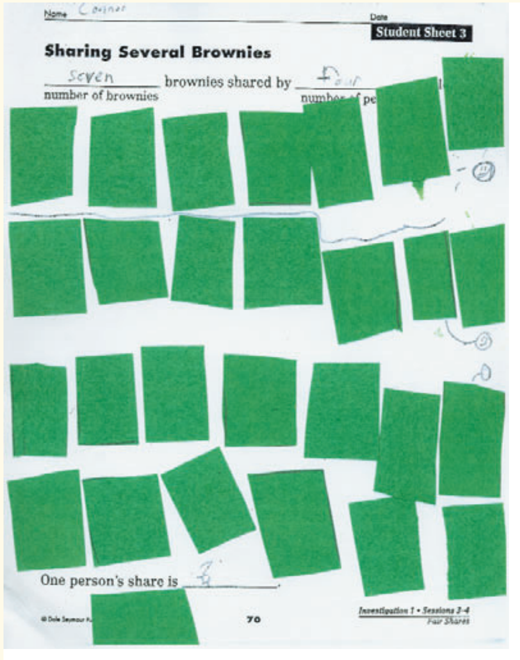
1 3/4 = 7/4
1 + 1/2 + 1/4 = 1 3/4
7/4 = 1 + 1/2 + 1/4

In this situation, students may be seeing the equal sign in a new way. Students often assume that "=" means that an operation on the left side of the sign must be performed and the result must be written on the right side. Understanding the meaning of the equal sign is important and will be crucial in students' mathematical future (in algebra, for example). Teachers in earlier grades can lay a foundation for this understanding by using the equal sign to show equivalency.

Even after students have realized that the answers 1 3/4, 1 + 1/2 + 1/4, and 7/4 represent the same amount, giving them other opportunities to make explicit why some fractions are equivalent is important. For example, when stating 1 3/4 = 1 + 1/2 + 1/4, they must make explicit that 1/2 + 1/4 = 3/4. Introducing the concept of common denominator in third grade is unnecessary; students can realize that the fractions are equivalent by overlapping the rectangles that represent them. Students will realize that 3/4 and 1/2 + 1/4 do indeed cover the same amount of the unit rectangle. They can also break 1/2 into two equal parts and use the fact that 1/2 = 1/4 + 1/4 to make the connection. In the same way, students can realize that 1 3/4 and 7/4 are equivalent.

Figure 4

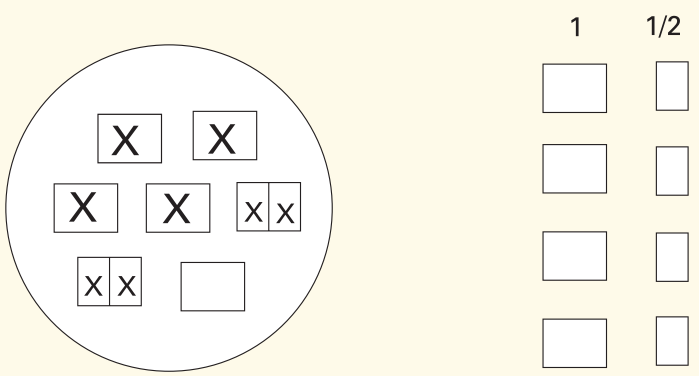
Connor's solution



Or they can use the fact that 1 = 4/4 to see that there are seven 1/4s in 1 3/4. At this age, developing efficient algorithms to change between mixed fractions and improper fractions is unimportant. What is important is that students have ways to convince themselves and others whether or not two fractions are indeed equivalent. When comparing fractions to establish equivalency, it may also be convenient or even necessary for the teacher to provide pre-cut

Figure 5

Crossing out brownies that have been shared



fraction pieces so that the comparisons are accurate.

Related to the concept of equivalent fractions is the idea of equal ratios. Students may realize that dividing two brownies among four people is the same as dividing one brownie between two people (see **fig. 6**). In both cases, each person receives the same amount. Some will notice an inverse relationship between the number of the pieces and their size for fractions that are equivalent. For example, when comparing numerical solutions such as 2/4 and 1/2, the first fraction has twice the number of pieces as the other but the pieces are only half as big; comparing 3/6 with 1/2, the number of pieces is three times more but the size is only one-third as much.

Students can solve problems such as 2 ÷ 4, 3 ÷ 6, and 4 ÷ 8 and realize that in each case the result is 1/2, or a fraction equivalent to 1/2 such as 2/4, 3/6, or 4/8. Some of the students will notice that the number to be divided is one-half of the divisor. Some will realize that the result of two divisions is the same if in both cases the ratio between dividend and divisor is the same. So 3 ÷ 2 is the same as 30 ÷ 20 and 300 ÷ 200. In the upper elementary grades, this realization will help students understand the reason why, when dividing decimals such as 2.5 ÷ 0.5, we can instead solve the problem 25 ÷ 5 and obtain the same answer.

In the same way, some of the students will see that the numerator is one-half of the denominator in fractions that are equivalent to 1/2. This realization will lay the foundation for the idea that fractions represent not only a quantity but also a ratio. Equivalent fractions are equal as quantities but also represent the same ratio.

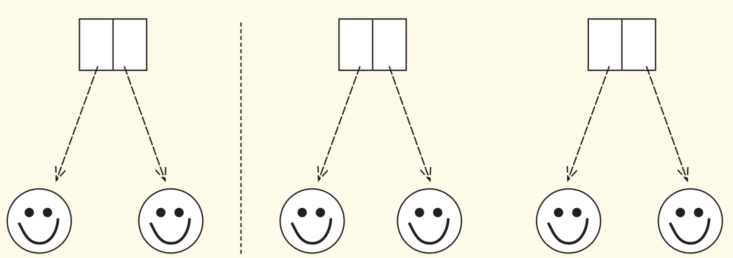
Another important connection in a division problem is the relationship between the corresponding fraction and the remainder and the divisor. For example, 9 ÷ 4 can be reported as 2 R1; it can also be reported as 2 1/4. Many students must be guided through the additional step of dividing the remainder to obtain a fractional answer. A context in which dividing the remainder makes sense can enhance the connection. When thinking about 9 ÷ 4 in terms of money, for example, students will realize that each person will receive two dollars and one quarter, or 2 1/4.

In Connor's approach described earlier, having seven pieces of size 1/4 can seem very natural. As Connor did, this can be written as 7/4. This fraction is clearly bigger than 4/4 or one unit. In mathematics, the word *fraction* denotes the quotient of two quantities, and it is not uncommon that the quantity being divided is bigger than the divisor. In everyday language, however, *fraction* usually is used to denote a fragment—a part, not the whole. Often, all the examples of fractions that students first encounter in school are parts of a unit. Furthermore, the unfortunate name *improper fraction* may suggest to some children that there is something "illegal" about a fraction such as 7/4 or that using such fractions is not proper. To facilitate dealing with fractions such as 7/4, it is important to let students view fractions such as 3/4 not only as three out of four pieces but also as three pieces of size 1/4 each. For many students, extending the latter meaning to fractions bigger than the unit is easier, whereas others have problems with the "seven out of four" interpretation.

Students in the upper elementary grades can make connections between fractions and area. When the unit fraction is a square, students can see that when the area of a rectangle represents a fraction, the length and width also represent fractions. For example, when one unit square is divided into six equal rectangles of area 1/6 as shown in **figure 7,** the length and width of the small rectangles will be 1/2 and 1/3 of the length of the side of the original square. Students can see that for fractions, too, the product of the sides of a rectangle renders the area of the rectangle, 1/2 × 1/3 = 1/6.

Figure 6

1 ÷ 2 gives the same as 2 ÷ 4



Addressing Misconceptions

Remembering that students do not come to school as blank slates is important. By the time they enter school, many students have developed a lot of informal mathematical ideas. In school they develop their own notions or conceptions of mathematical ideas based on the examples and experiences that their textbooks and teachers provide. Sometimes the conceptions that children form are based on a limited number of examples; sometimes children form conceptions that the teacher did not

intend. These conceptions are often incomplete or even erroneous. It is also important to remember that these preconceptions can be quite resilient, and for the teacher to show the "right way" is not enough.

Students will have previous conceptions about division. For example, one of the students in Ms. Klein's classroom objected to the division problem 7/4, or 7 ÷ 4, because 4 does not go evenly into 7. The teacher may guide students to make sense of the measurement interpretation of division and of "How many times does 4 go into 7?" For some students, concrete representations of 7 and 4 that lend themselves to measuring one quantity in terms of the other, such as strips of paper with clearly marked segments, can be helpful. Students can see that 4 does not go into 7 two times, but it does go in one time, and some room is still left over. Students can see that they can fit three out of the four segments that form 4, so that the answer is 1 3/4.

Even when students have the concrete representation of fractions in front of them, they commonly operate only with the symbolic representation and forget to check whether or not the answer makes sense. A common mistake when trying to determine 1/2 + 1/4 is to "add across" and obtain an answer of 2/6. For the teacher to point out that fractions are not added that way is not enough. Instead, the student could represent the problem with a piece of 1/2 and a piece of 1/4, explain what it means to add the two quantities, and realize that the answer will be bigger than 1/2. The student can also represent 2/6 of the same unit and see that this fraction is less than 1/2, so that 2/6 cannot be the answer to the problem 1/2 + 1/4.

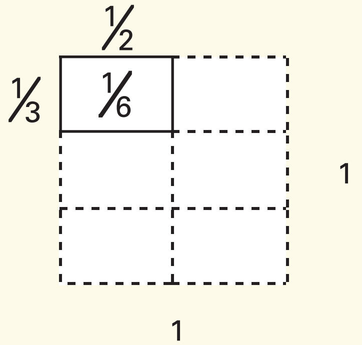
Relating the problem to more familiar problems can help students make sense of the answer. In the beginning, students make sense mainly through connections to concrete objects and situations. Gradually, students should develop the ability to make sense by relating a problem in mathematics to other mathematical situations with which they are more familiar. For example, when asked how he obtained the answer 1 3/4 for the division problem 7 ÷ 4, Bokú, a student in the upper elementary grades, responded, "Four times 1 1/2 is six; four times two is eight; so the answer has to be halfway between 1 1/2 and 2."

Conclusion

Given the proper context, tools, and guidance, young students can develop an understanding of fractions and make connections to other areas of mathematics. With teachers' help, they can build a foundation to later deal in more systematic ways with equivalent fractions, converting fractions from one representation to another and moving back and forth between division of whole numbers and rational numbers. Of course, it is similarly important that teachers have a good grasp of the concepts related to fractions. As illustrated above, it is of equal importance however, that teachers have the knowledge of fractions most relevant and fitting to their teachability. According to Shulman (1986), this knowledge includes "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations" (p. 9). Not all examples or contexts are equally powerful to establish connections, nor do all serve equally well to dispel misconceptions. That no single most powerful form of presenting an idea exists is clear from the variety of strategies that students use. Teachers must develop a repertoire of alternative approaches and forms of presenting ideas. Teachers can derive these from several sources, including their students' strategies, research, exemplary materials, other teachers, and reflection on their own experiences of what works and why.

Figure 7

The area of the rectangle is length times width



References

Shulman, Lee S. "Those Who Understand: Knowledge Growth in Teaching." *Educational Researcher* 15 (1) (1986): 4–14.

Tierney, Cornelia, and Mary Berle-Carman. *Fair Shares.* Palo Alto, Calif.: Dale Seymour Publications, 1995.

Toluk, Zulbiye. *Children's Conceptualization of the Quotient Subconstruct of Rational Numbers.* Unpublished doctoral diss., Arizona State University, 1999.